Uniform Convergence

$S \subset \mathbb{R}$

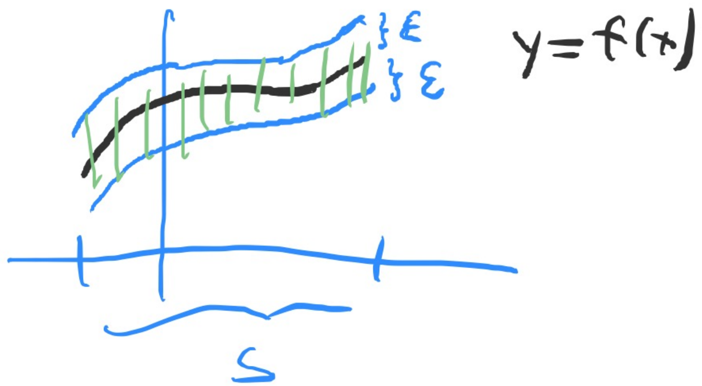
functions $f_n : S \to \mathbb{R}$

$f : S \to \mathbb{R}$

Def The functions f_n converge uniformly to f if for every $\varepsilon > 0$ we can find N s.t.

$|f_n(x) - f(x)| < \varepsilon$ for all $x \in S$, for all $n \geq N$

informally:



can find $N \in \mathbb{N}$ s.t. the graphs of all f_n's are in green 2ε strip around $y = f(x)$ for $n \geq N$

have seen: $S = [0,1]$

$f_n(x) = x^n$

$\Longrightarrow$ f_n converge pointwise to $f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$

but they do NOT converge uniformly

Theorem Assume $f_n \to f$ uniformly
and all f_n's are continuous at $x_0 \in S$
$\Longrightarrow$ f is continuous at x_0

Proof have to estimate $|f(x) - f(x_0)|$ for x near x_0

idea: use properties of f_n's:

$$|f(x) - f(x_0)| = |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)|$$

$$\leq \underbrace{|f(x) - f_n(x)|}_{\text{small for large } n} + \underbrace{|f_n(x) - f_n(x_0)|}_{\text{small for } x \text{ near } x_0 \text{ because } f_n \text{ cont.}} + \underbrace{|f_n(x_0) - f(x_0)|}_{\text{small for large } n}$$

formal proof:

pick $\varepsilon > 0$

$$|f(x)-f(x_0)| \le \underbrace{|f(x)-f_N(x)|}_{<\varepsilon/3 \text{ if } n\ge N} + \underbrace{|f_N(x)-f_N(x_0)|}_{<\varepsilon/3 \text{ if } |x-x_0|<\delta} + \underbrace{|f_N(x_0)-f(x_0)|}_{<\varepsilon/3 \text{ if } n \ge N}$$

pick $n \ge N$

Want each of the summands to be less than $\varepsilon/3$

- can find N s.t. $|f(x)-f_n(x)| < \varepsilon/3$ for all $n \ge N$ for all $x \in S$ (including x_0)

- f_N is continuous

$\Rightarrow$ can find $\delta > 0$ s.t.

$|f_N(x) - f_N(x_0)| < \varepsilon/3$ if $|x-x_0| < \delta$

$\Rightarrow$ if $|x-x_0| < \delta$ then

$|f(x)-f(x_0)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$

$\Rightarrow$ f cont. at x_0.

we will see later: can be used to prove that power series are continuous if $|x| < R$

radius of convergence

<u>Remark:</u> If $f_n \to f$ uniformly on S

$\Rightarrow$ $f_n(x) \to f(x)$ for all $x \in S$, i.e. $f_n \to f$ pointwise.

$\Rightarrow$ pointwise convergence necessary for uniform convergence

<u>Examples</u> ① let $f_n(x) = \dfrac{x}{1+nx^2}$

Do the f_n's converge pointwise? uniformly?

<u>Sol.</u> first check pointwise convergence!

<u>$x=0$:</u> $f_n(0) = \dfrac{0}{1+0} = 0$ for all n.

$\lim_{n\to\infty} f_n(0) = 0$

$$f_n(x) = \frac{x}{1+nx^2}$$

$$\boxed{f_n\left(\sqrt{1/n}\right) = \frac{\sqrt{1/n}}{1+n\cdot\frac{1}{n}} = \frac{1}{2}\sqrt{1/n}}$$

$x \neq 0$ can treat x as a constant.

$$f_n(x) = \frac{1}{\underbrace{1/x + nx}_{\to\infty \text{ for } n\to\infty}} \longrightarrow 0 \quad \text{for } n \to \infty$$

$\Rightarrow$ $f_n(x) \longrightarrow f(x)=0$ pointwise! $f=$ zero function

uniform convergence?

for any $\varepsilon>0$ need to find N s.t. $|f_n(x)| < \varepsilon \ \forall x \in \mathbb{R}$

need to find max/min of $f_n(x)$.

$$0 = f_n'(x) = \frac{(1+nx^2)\cdot 1 - 2nx\cdot x}{(1+nx^2)^2}$$

$\Rightarrow$ $0 = 1+nx^2 - 2nx^2$ $\Rightarrow$ $1 = nx^2$ $\boxed{x = \pm\sqrt{1/n}}$

$\Rightarrow \quad |f_n(x)| \leq f(1/\sqrt{n}) = \frac{1}{2}\sqrt{1/n}$.

pick $\varepsilon > 0$ want $\frac{1}{2}\sqrt{1/n} < \varepsilon \iff \sqrt{1/n} < 2\varepsilon$

$\iff n > \frac{1}{4\varepsilon^2}$

pick any $\boxed{N > \frac{1}{4\varepsilon^2}}$ $n \geq N$

$\Rightarrow |f_n(x) - 0| \leq \frac{1}{2}\sqrt{1/n} \leq \frac{1}{2}\sqrt{1/N} < \varepsilon$

Result: $f_n \to f$ = zero function uniformly.

(2) same question for $\boxed{f_n(x) = n^2 x^n (1-x)}$, $S = [0,1]$

again: $f_n(0) = 0 \ \forall n \to f_n(0) \to 0$

same $f_n(1) = 0 \ \forall n \to f_n(1) \to 0$

$0 < x < 1$: $|f_n(x)| = |n^2 x^n (1-x)| \leq n^2 x^n \to 0$ check!

use: if $\lim \left|\frac{a_{n+1}}{a_n}\right| = s < 1 \Rightarrow \lim a_n = 0$: if $a_n = n^2 x^n \Rightarrow \lim \left|\frac{a_{n+1}}{a_n}\right| = x < 1$

recall: $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$

pointwise convergence

uniform convergence?

need to determine max/min of f_n:

$$0 = f_n'(x) = n^2 (n x^{n-1}(1-x) - x^n)$$

$$\Rightarrow n(1-x) = x$$

$$\Rightarrow n = (n+1)x$$

$$\Rightarrow \boxed{x = \frac{n}{n+1}}$$

obtain max. at $x = \frac{n}{n+1}$

$$f_n\left(\frac{n}{n+1}\right) = n^2 \cdot \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right)$$

$$= n^2 \left(\frac{1}{(n+1)/n}\right)^n \frac{1}{n+1} = n \cdot \frac{1}{(1+\frac{1}{n})^n} \cdot \frac{n}{n+1}$$

again: $f_n\left(\frac{n}{n+1}\right) = \underbrace{n}_{\downarrow \atop \infty} \cdot \underbrace{\frac{1}{\left(1+\frac{1}{n}\right)^n}}_{\frac{1}{e}} \cdot \underbrace{\frac{n}{n+1}}_{\downarrow \atop 1}$ for $n \to \infty$

uniform convergence to 0?

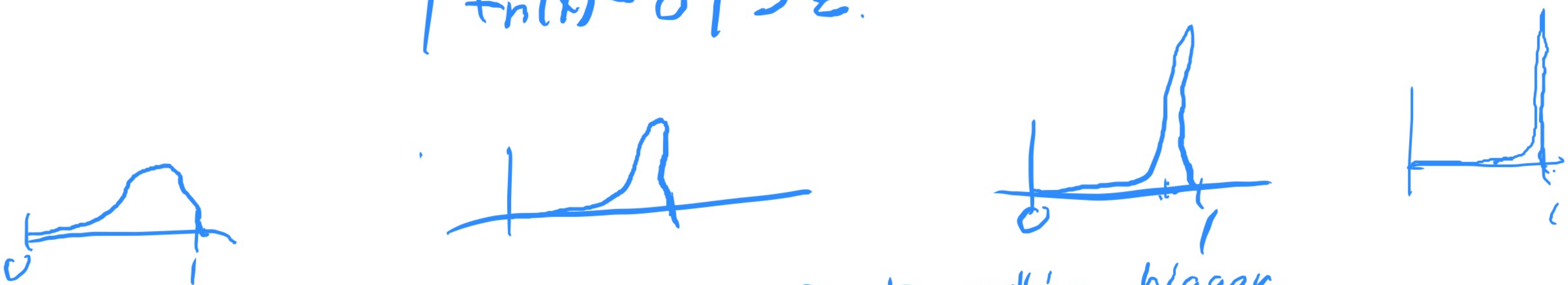
we observe:

max of f_n on $[0,1]$ goes to ∞.

$\Rightarrow$ no uniform convergence

(for any $\varepsilon > 0$ we can find an x s.t.

$$|f_n(x) - 0| > \varepsilon.$$
)



n incre $\longrightarrow$ n getting bigger.